ON QUASIBRITTLE FRACTURE

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The Griffith theory of brittle fracture is extended to plastic materials by the concept of quasibrittle fracture advanced by Irwin [1] and Orowan [2]. Here, instead of the true surface energy which figures in the Griffith theory, an effective surface energy density is taken which is equal to the irreversible work of plastic deformation in a thin layer near the surface of the crack plus the true surface energy per unit area [1–5].

In the present article it is shown that the magnitude of the ratio of the true surface energy to the irreversible work of plastic deformation is of the order of σ_s / E, where σ_s is the yield stress in tension and E is Young's modulus. The investigation is based on the solution of an elastic-plastic problem for a plate with a crack on a general approach to crack propagation in an arbitrary continuous medium [3]. The elastic-plastic analogue of Griffith's problem is also considered here. It turns out that, unlike the situation for brittle materials, a crack in an elastic-plastic material begins to grow stably as stress increases, and only when the load reaches some critical value does instability ensue. These qualitative pecularities of crack development in elastic-plastic materials are well known (e.g. for metals [4 and 5]). The solution of the problem referred to above permits estimation of the limit of applicability of the notion of quasibrittle fracture.

1. An elastic-plastic problem for a plate with a slit. Let us examine a thin plate having an arbitrary tension crack and subjected to the action of tensile forces. We shall consider the material of the plate to be elastic-perfectly plastic, satisfying the Tesca-Saint-Venant yield condition. We introduce a system of rectangular cartesian coordinates x y in the plane of the plate with origin O at the tip of the crack, the x-axis being directed along the crack (Fig. 1). We shall examine a neighborhood of the tip of the crack which is small relative to the characteristic linear dimension of the plate, but large relative to the characteristic linear dimension of the plastic region. We shall assume that a condition of local symmetry is satisfied. The crack is represented in the xy plane by a semi-infinite slit along the negative half of the x-axis, the edges of the slit being free of traction (Fig. 1). We recall that the maximum shearing stress at every point in an elastic-plastic body with a Tresca yield condition cannot exceed the yield stress in shear τ_s $(2\tau_s = \sigma_s)$.

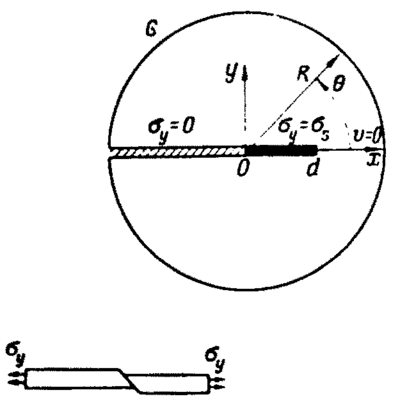

Fig. 1

We shall show that the solution of the elastic-plastic problem which has been posed in Dugdale's formulation is expressed by the following formulas:

$$\sigma_x + \sigma_y = 2\sigma_s\left(1 - \frac{1}{\pi}\arg\frac{i\sqrt{d} - \sqrt{z-d}}{i\sqrt{d} + \sqrt{z-d}}\right)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = \frac{2i\sigma_s y\sqrt{d}}{\pi z\sqrt{z-d}} \qquad (z = x + iy) \tag{1.1}$$

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{1-\nu}{1+\nu}\sigma_s\left(1 - \frac{1}{\pi}\arg\frac{i\sqrt{d} - \sqrt{z-d}}{i\sqrt{d} + \sqrt{z-d}}\right) +$$

$$+ \frac{2\sigma_s}{\pi i(1+\nu)}\ln\left|\frac{i\sqrt{d} - \sqrt{z-d}}{i\sqrt{d} + \sqrt{z-d}}\right| + \frac{i\sigma_s\sqrt{d}y}{\pi z\sqrt{z-d}}$$

Here σ_x, σ_y and τ_{xy} are the components of the stress tensor; u and v are the components of the displacement vector in the x and y directions; μ is the shear modulus; ν is Poisson's ratio; and $\arg f$ is the argument of the complex number $f\,(-\pi \leqslant \arg f \leqslant \pi)$. The function $\sqrt{z-d}$ is analytic outside the semi-infinite cut $x < d$ along the real axis and is positive on the continuation of the cut $x > d$.

The plastic region is a segment of length d on the continuation of the crack along the x-axis (Fig. 1). The width of the region is zero. In thin plates this can physically be realized in the form of a slip plane making an angle of 45° with the plane of the plate. In thick plates, characteristic thickened regions remain on the continuation of the crack, giving a grooved effect (Fig. 1). The stresses in the plastic region are

$$\sigma_x = \sigma_y = \sigma_s, \qquad \tau_{xy} = 0. \tag{1.2}$$

Equations (1.1) correspond to a choice of the Muskhelishvili potentials $\Phi(z)$ and $\Psi(z)$ in the form [6]

$$\Phi(z) = \frac{1}{2}\sigma_s\left(1 - \frac{1}{\pi i}\ln\frac{i\sqrt{d} - \sqrt{z-d}}{i\sqrt{d} + \sqrt{z-d}}\right) \qquad \Psi(z) = -z\Phi'(z) \tag{1.3}$$

As is easily seen, the equations of the theory of elasticity and the boundary conditions are satisfied outside the cut $y = 0$, $x < d$.

In the problem under consideration, the stress intensity factor N is the loading parameter. This quantity specifies the distribution of stresses and displacements at the point at infinity (i.e. at distances which are large compared to d, but small in relation to the characteristic linear dimension of the body).

It should be noted that for plane stress, plates made of elastic-perfectly plastic material generally exhibit a characteristic tendency for formation of the plastic zone in narrow regions of slip. Thus, for example, according to the exact solution of the elastic-plastic problem of biaxial tension in a plate with a circular hole [7], the plastic zone goes from a circular region to an elongated one with a width-to-length ratio of 1 : 4 for a change in the far field from a hydrostatic state by as little as 0.1 ($\Delta\sigma / \sigma \simeq 0.1$).

According to the solution due to Muskhelishvili, the elastic stress and displacement fields near the crack tip are given by the following expressions:

$$\sigma_x + \sigma_y = \frac{2N}{\sqrt{r}}\cos\frac{\theta}{2}, \qquad \sigma_x - i\tau_{xy} = \frac{N}{4\sqrt{r}}\left(e^{-1/_2 i\theta} + 2e^{1/_2 i\theta} + e^{-3/_2 i\theta}\right) \tag{1.4}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{(1+\nu)N}{4E\sqrt{r}}\left(2\frac{3-\nu}{1+\nu}e^{-1/_2 i\theta} - 3e^{-1/_2 i\theta} + e^{3/_2 i\theta}\right)$$

$$\Phi(z) = \frac{N}{2\sqrt{z}}, \qquad \Psi(z) = -z\Phi'(z)$$

where r and θ are polar coordinates with origin at the crack tip ($\theta = 0$ corresponds to the continuation of the crack). The expressions (1.4) are also obtained from Eqs. (1.1) and (1.3) as $z \to \infty$, in which case the stress intensity factor then turns out to be equal to

$$N = \frac{2}{\pi} \sigma_s \sqrt{d} \tag{1.5}$$

Therefore, the size of the plastic region d is found in accordance with Eq. (1.5) from the solution of a purely elastic problem; it is completely determined by the stress intensity factor.

The displacement v of the edges of the slit $y = 0$; $x < d$ are found by Eqs. (1.1)

$$v = \pm \frac{2\sigma_s}{\pi E} \int_x^d \ln \left| \frac{\sqrt{d} - \sqrt{d-x}}{\sqrt{d} + \sqrt{d-x}} \right| dx =$$

$$= \pm \frac{2\sigma_s}{\pi E} \left[2\sqrt{d(d-x)} + x \ln \left| \frac{\sqrt{d} - \sqrt{d-x}}{\sqrt{d} + \sqrt{d-x}} \right| \right] \tag{1.6}$$

The displacement v_0 of the crack tip is then equal to

$$v_0 = \pm \frac{4\sigma_s}{\pi E} d = \pm \frac{\pi N^2}{\sigma_s E} \tag{1.7}$$

The reasoning presented above can be extended in a completely analogous way to the general case of an arbitrary number of cracks located along a single straight line in an infinite plate, if the only applied loads are normal to the edges of the crack.

The solution of this class of elastic-plastic problems is obtained by the method of Muskhelishvili [6]. The linear dimensions of the zones are determined from the conditions of solvability of the boundary value problem. Of course, in this case under consideration, these zones need not necessarily be small with respect to the characteristic linear dimension of the plate. In carrying this out, it is necessary to insure that the condition $|\sigma_1 - \sigma_2| > \sigma_s$ is still satisfied in the elastic region. For some values of the loading parameters this relation will cease to hold. Secondary plastic regions then develop in which slip proceeds along planes normal to the plane of the plate.

Analogous solutions of the elastic-plastic problem for a crack of finite length in a plate were first obtained by Dugdale [8]. The same problem was later investigated much more thoroughly in paper [9].

2. The energy equation. We shall use the elastic-plastic solution (1.1) for the analysis of the development of a cleavage crack in a plate made of an elastic-plastic material. Here we shall apply the general approach proposed in [3] which is based on the law of conservation of energy in the medium near a small neighborhood of the crack tip. As regards local fracture processes and deformations near the crack tip which determine the crack growth as a whole, we shall assume only that they are accompanied by a certain energy absorption in the formation of the new surface of the crack.

The general condition of limiting equilibrium of the boundary of the crack which was introduced in [3] for an arbitrary continuous medium (Eq. (1.15) of [3]) may be written in the following form for the case under consideration:

$$\lim \left[\oint_C (\mathcal{E} \cos\theta - A_*) \, ds \right] = 2\gamma_* \tag{2.1}$$

$$\mathcal{E} = \int \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + 2\tau_{xy} d\varepsilon_{xy} = \frac{1}{2} E^{-1} [\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2]$$

$$A_* = (\sigma_x \cos\theta + \tau_{xy}\sin\theta)\frac{\partial u}{\partial x} + (\tau_{xy}\cos\theta + \sigma_y \sin\theta)\frac{\partial v}{\partial x}$$

where ε_x, ε_y and ε_{xy} are the strain components, R is the radius of the circle C (Fig. 1). The magnitude of the effective surface energy γ_* per unit surface of crack formed is equal to the work of plastic deformation on the area of the slip, γ_p, plus the true surface energy γ [1–5].

The integral in Eq. (2.1) is calculated just as in [3]. The distributions of displacements and stresses for large R are given by Eqs. (1.4). Finally, we obtain

$$N^2 = \pi^{-1}E\gamma_* \tag{2.2}$$

This is Irwin's formula for plane stress, which was obtained in a different way [10].

Now let us calculate γ_p using Eqs. (2.2), (2.1) and (1.7)

$$\gamma_p = \int_0^{r_0} \sigma_y\, dv + \sigma_s v_0 O\left(\frac{\sigma_s}{E}\right) = \gamma_* + \gamma_* O\left(\frac{\sigma_s}{E}\right) \tag{2.3}$$

From this we obtain the desired relation between γ_* and γ

$$\gamma = \gamma_* O(\sigma_s/E) \tag{2.4}$$

We note that physically the quantity

$$\int_0^{r_0} \sigma_y\, dv$$

is the resultant of the external forces applied to the plastic layer $(0, d)$ and directed along the x-axis. This justifies the force interpretation of γ_* as some effective surface tension of the material applied to the end of the crack.

Thus the dimension of the plastic region ahead of the crack and the effective surface energy are completely determined by the constants of the material E, σ_s, γ and ν. For most metals, and in particular for structural steels, the magnitude of E/σ_s is of the order of 10^3. Therefore, Eq. (2.4) gives the relation $\gamma_* \approx 10^3\gamma$, which, in order of magnitude, is well supported by experiments [11–14].

In processes where intragranular microcracks initiate in steels, the quantity σ_s should correspond to the theoretical strength, which is of the order of $0.1\,E$.

Here the basic equation (2.4) gives the magnitude of the effective surface energy which characterizes the early stage of crack initiation as approximately 10γ, i.e. for steels, about 2×10^4 erg/cm^2. The numbers obtained agree well with Cottrell's dislocation theory and with the experimental data [14].

The very large values of γ_* for plastic steels, which reach 10^8 erg/cm^2 are explained on the basis of the preceding comments by the fact that the quantity γ_* also includes the irreversible work in the main mass of the material, e.g. in secondary plastic zones.

As an example, let us calculate the values of v_0 and d for a low-carbon steel using the data of [11], $\gamma_* = 2 \times 10^6$ dyne/cm, $\sigma_s \approx 4 \times 10^9$ dyne/cm^2, $E = 2 \times 10^{12}$ dyne/cm^2. From Eqs. (1.5), (1.7) and (2.2), we obtain

$$r_0 = \gamma_* / \sigma_s = 5 \cdot 10^{-4} \text{ cm}, \qquad d = (\pi E \gamma_*)/(4\sigma_s^2) = 0.2 \text{ cm} \tag{2.7}$$

Equation (2.4) permits us to treat the concept of quasibrittle fracture and the adsorption effect from a unified point of view. It should be remarked that the idea of relating γ_* to γ was apparently first expressed by Gilman [15]; however, the relationship obtained by him on the basis of rough estimates is not verified by experiment.

Under suitable assumptions Eq. (2.4) can also be obtained by dimensional analysis.

Note. The expression (2.3) is written correct to first-order quantities; while Eq. (2.1) is also valid for finite deformations of the elastic medium if $\mathcal{E}$ denotes the strain energy. This may be verified easily if the calculations of Sect. 1 of [3] are checked and the local law of conservation of energy is used for finite deformations of the medium [16].

Since the derivatives of the displacements with respect to the coordinates go to zero as $R \to \infty$ in the singular solution (1.4), Irwin's formula (2.2) is also valid for finite deformations of an elastic-perfectly plastic solid in a small neighborhood of the crack contour.

3. The elastic-plastic analogue of Griffith's problem. We shall consider a plate made of an elastic-perfectly plastic material with a straight crack through its thickness. The crack has a length $2l$ and is in a homogeneous field of tensile stress $\sigma_y = \sigma$. We take the edges of the crack to be traction free. We shall use rectangular cartesian coordinates x, y with origin at the center of the crack, the x-axis being directed along the crack.

The plastic regions near the ends of the crack will be segments of length d along the continuations of the crack. The solution of the boundary value problem of the two-dimensional theory of elasticity

$$\begin{gathered}\tau_{xy} = 0 \quad \text{for } y = 0, \qquad \sigma_y = 0 \quad \text{for } y = 0 \ |x| < l \\ \sigma_y = \sigma_s \quad \text{for } y = 0 \ l < |x| < l + d \\ v = 0 \quad \text{for } y = 0 \ |x| > l + d, \qquad \sigma_y \to \sigma \quad \text{for } |x + iy| \to \infty\end{gathered} \tag{3.1}$$

is obtained by the Kolosov-Muskhelishvili method [8 and 9]. This solution can be written as

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z) \quad (z = x + iy) \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = \sigma - 4iy\Phi'(z)$$

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{3 - \nu}{1 + \nu}\Phi(z) - \overline{\Phi(z)} - \frac{1}{2}\sigma - 2iy\overline{\Phi'(z)} \tag{3.2}$$

$$\Phi(z) = -\frac{\sigma}{4} - \frac{\sigma_s}{2\pi i}\ln\frac{l - z}{l + z} -$$

$$- \frac{\sigma_s}{2\pi i}\ln\frac{\sqrt{d(2l + d)}\sqrt{(l + d)^2 - z^2} + lz + (l + d)^2}{\sqrt{d(2l + d)}\sqrt{(l + d)^2 - z^2} - lz + (l + d)^2}$$

$$\frac{d}{l} = \sec\frac{\pi\sigma}{2\sigma_s} - 1$$

The radical $\sqrt{(l + d)^2 - z^2}$ is taken as positive on the upper edge of the cut $(-l - d, l + d)$ along the x-axis, and $\ln F = \ln|F| + i\arg F \ (-\pi < \arg F < \pi)$

Using (3.2) we calculate the displacements u and v for $y = 0$, $l < |x| < l + d$:

$$u = u(x, l, \beta) = \frac{2\sigma_s}{\pi E}\left[\frac{\pi}{2}(1 - \nu) - \beta\right]x - \frac{\sigma_s l}{E}(1 - \nu) \qquad \left(\beta = \frac{\pi\sigma}{2\sigma_s}\right)$$

$$v = v(x, l, \beta) = \pm\frac{2\sigma_s l}{\pi E}\Big[-\ln(t^2 - 1) + 2\ln\left(\sin\beta + \sqrt{1 - t^2\cos^2\beta}\right) -$$

$$- 2\ln\cos\beta + t\ln\frac{(t - 1)\left(\sin\beta\sqrt{1 - t^2\cos^2\beta} + t\cos^2\beta + 1\right)}{(t + 1)\left(\sin\beta\sqrt{1 - t^2\cos^2\beta} - t\cos^2\beta + 1\right)}\Big] \left(t = \frac{x}{l}\right) \tag{3.3}$$

$$\text{for } |x| < l, \quad u = -\sigma x / E$$

The solution (3.2) depends on a single undetermined parameter l. In order to find it we invoke the law of conservation of energy and a physical representation of the fracture energy which is completely analogous to the Griffith-Irwin-Orowan concepts. At some instant of time let the corresponding values of the parameters l and β, the discontinuity of the v displacement on the interval $(l, l + d)$, be represented schematically by the triangle $A_0C_0B_0$ (Fig. 2). After some time which corresponds to increments Δl and $\Delta\beta$ in the parameters l and β the segment under study will occupy a position which is shifted in the direction of crack growth and is rêpresented by the triangle $A_1C_1B_1$. The points C_0 and C_1 correspond to the ends of the plastic zones; the points A_0, A_1 and B_0, B_1 are the ends of the crack on the two sides of the plastic line of discontinuity.

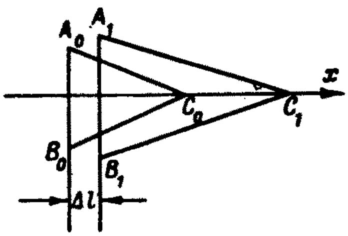

Fig. 2

In accordance with (3.2), the stresses in the plastic regions are

$$\sigma_y = \sigma_s, \qquad \sigma_x = \sigma_s - \sigma, \qquad \tau_{xy} = 0 \tag{3.4}$$

As the crack tip advances by Δl, the stress σ_y does work on the corresponding displacement. This work is obviously equal to the energy dissipation $\gamma_* \Delta l$. The fracture energy γ_* is considered to be a constant characteristic of the material. We shall not consider terms of higher order.

$$\gamma_* \Delta l = \int_{l+\Delta l}^{l+d} \sigma_y [v(x, l + \Delta l, \beta + \Delta\beta) - v(x, l, \beta)]\, dx + \Delta S \tag{3.5}$$

For small Δl the following estimates are valid:

$$\int_{l}^{l+\Delta l} \sigma_y [v(x, l + \Delta l, \beta + \beta\Delta) - v(x, l, \beta)]\, dx \approx \left(\sigma_y \frac{\partial v}{\partial l}\right)_{x=l} (\Delta l)^2$$

$$\int_{l+d}^{l+d+\Delta d} \sigma_y [v(x, l + \Delta l, \beta + \Delta\beta) - v(x, l, \beta)]\, dx \approx \left(\sigma_y \frac{\partial v}{\partial l}\right)_{x=l+d} (\Delta l)^2$$

$$(\Delta S \approx O[(\Delta l)^2]) \tag{3.6}$$

We expand the function $v(x, l + \Delta l, \beta + \Delta\beta)$ in a Taylor series in powers of Δl in Eq. (3.5) and let $\Delta l \to 0$, taking the estimates (3.6) into account. We then obtain the following expression:

$$\gamma_* = \sigma_s \int_{l}^{l+d} \left(\frac{\partial v}{\partial l} + \frac{\partial v}{d\beta}\frac{\partial \beta}{\partial l}\right) dx \tag{3.7}$$

In particular, for $d \ll l$ (when $\beta \to 0$) $\partial u/\partial\beta$ and $\partial v/\partial\beta$ will approach zero and the condition of stationary character of the crack, $\partial v/\partial l = \partial v/\partial x$ will hold. Equation (3.7) then becomes identical with (2.2). Using (3.2), we compute

$$\int x \frac{\partial v}{\partial x}\, dx = \frac{2\sigma_s l^2}{\pi E} [\operatorname{tg}\beta \arccos(t\cos\beta) +$$

$$+ \frac{1}{2}(t^2 - 1)\ln\frac{(t-1)(\sin\beta\sqrt{1 - t^2\cos^2\beta} + t\cos^2\beta + 1)}{(t+1)(\sin\beta\sqrt{1 - t^2\cos^2\beta} - t\cos^2\beta + 1)}\Bigg]\left(t = \frac{x}{l}\right) \tag{3.8}$$

With the aid of (3.3), (3.4) and (3.8), we find

$$\sigma_s \int_l^{l+d} \left(\frac{\partial v}{\partial l} + \frac{\partial v}{\partial \beta}\frac{d\beta}{dl}\right) dx = \sigma_s \frac{d}{dl} \int_l^{l+d} v(x, l, \beta)\, dx + \sigma_s v(l, l, \beta) =$$

$$= -\sigma_s \frac{d}{dl} \int_l^{l+d} x \frac{\partial v}{\partial x}\, dx - \sigma_s l \frac{dv(l, l, \beta)}{dl} = \tag{3.9}$$

$$= \frac{2\sigma_s^2 l}{\pi E}\left[2(\ln\cos\beta + \beta\,\mathrm{tg}\,\beta) + l(\beta\sec^2\beta - \mathrm{tg}\,\beta)\frac{d\beta}{dl}\right]$$

The method of calculation of γ_* which has been used is analogous to Irwin's method [10]. We shall now show how γ_* may be found by applying Griffith's method.

Let us consider an elastic medium occupying the fixed region $|z| \leqslant R$ where the radius R is large compared with the crack length. The state of this system is determined by two parameters: l and σ. According to the law of conservation of energy, the rate of doing external work by the external forces applied to the outer contour $|z| = R$ is equal to the rate of change of the elastic energy plus the rate of dissipation of energy. The parameter l can serve as a time-like parameter. We have

$$\frac{\delta A}{\delta l} = \frac{dW}{dl} + 4\gamma_* \tag{3.10}$$

The distribution of displacements and stresses for large R are found, in accordance with (3.2), by the following Muskhelishvili potentials

$$\Phi(z) = \frac{\sigma}{4} + \frac{\sigma_s l^2\,\mathrm{tg}\,\beta}{2\pi z^2} + O(z^{-4}) \quad \text{for} \quad z \to \infty \tag{3.11}$$

The calculations are carried out with the aid of Eqs. (3.2), (3.3) and (3.11), and also using Clapeyron's theorem; we obtain

$$\frac{\delta A}{\delta l} = R \int_0^{2\pi} \left[(\sigma_x\cos\theta + \tau_{xy}\sin\theta)\frac{du}{dl} + (\tau_{xy}\cos\theta + \sigma_y\sin\theta)\frac{dv}{dl}\right] d\theta =$$

$$= \frac{4\sigma_s^2 R^2}{\pi E}\beta\frac{d\beta}{dl} + \frac{\sigma_s^2 l}{\pi E}\left\{2(5+\nu)\beta\,\mathrm{tg}\,\beta + l\frac{d\beta}{dl}[(5+\nu)\beta\sec^2\beta - (3-\nu)\,\mathrm{tg}\,\beta]\right\}$$

$$W = \frac{1}{2} R \int_0^{2\pi} [(\sigma_x\cos\theta + \tau_{xy}\sin\theta)u + (\tau_{xy}\cos\theta + \sigma_y\sin\theta)v]\, d\theta - \tag{3.12}$$

$$-2\sigma_s \int_l^{l+d} v(x, l, \beta)\, dx = \frac{\pi\sigma^2 R^2}{2E} + \frac{\sigma_s^2 l^2}{\pi E}[-8\ln\cos\beta + (\nu - 3)\beta\,\mathrm{tg}\,\beta]$$

Considering l as the independent parameter, we easily find from Eqs. (3.10) and (3.12)

$$\gamma_* = \frac{2\sigma_s^2 l}{\pi E}\left[2(\ln\cos\beta + \beta\,\mathrm{tg}\,\beta) + l(\beta\sec^2\beta - \mathrm{tg}\,\beta)\frac{d\beta}{dl}\right] \tag{3.13}$$

Introducing the dimensionless length λ

$$\lambda = \frac{2\sigma_s^2 l}{\pi E \gamma_*} \qquad \left(\beta = \frac{\pi\sigma}{2\sigma_s}\right) \tag{3.14}$$

we finally write the desired relation between β and λ in the form of the following first-order differential equation:

$$\frac{d\beta}{d\lambda} = \frac{1 - 2\lambda(\ln\cos\beta + \beta\,\mathrm{tg}\,\beta)}{\lambda^2(\beta\sec^2\beta - \mathrm{tg}\,\beta)} \tag{3.15}$$

The field of the integral curves of Eq. (3.15) in the region $0 < \lambda < \infty$, $0 < \beta < \pi/2$ is shown in Fig. 3 (the calculations were carried out by G. D. Daianys on the

"NAIRI" computer). The stable parts of the curves are shown by the full lines, the unstable portions by the dashed lines. The curve determined by the equation

$$1 = 2\lambda\,(\ln\cos\beta + \beta\,\mathrm{tg}\,\beta) \qquad (3.16)$$

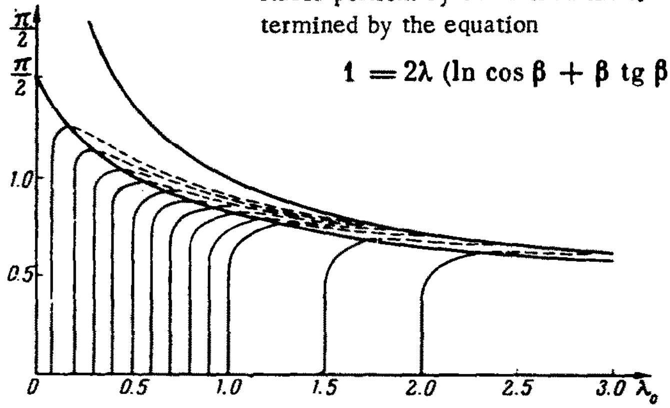

Fig. 3

divides the entire region of variation of the variables into a region of stable crack growth from some original crack $(d\beta / d\lambda > 0)$ and a region of instability, in which $d\beta / d\lambda < 0$.

Thus, a crack initially grows monotonically with increase in the load, attaining a maximum load at the point of intersection with the curve (3. 16), after which it becomes unstable. It is easy to show that in the region of instability all the integral curves approach the Griffith-Irwin-Orowan curve

$$\lambda\beta^2 = 1 \qquad \left(\sigma = \sqrt{\frac{2E\gamma_*}{\pi l}}\right) \qquad (3.17)$$

asymptotically for large λ.

The curves (3. 16) and (3. 17) are shown by the heavy lines in Fig. 3. The graph of the limiting load β_* versus the dimensionless length of the initial crack λ_0 is given in Fig.4. This curve was obtained from Fig. 3. For comparison, the curve of Eq. (3. 17) is also shown on the same figure.

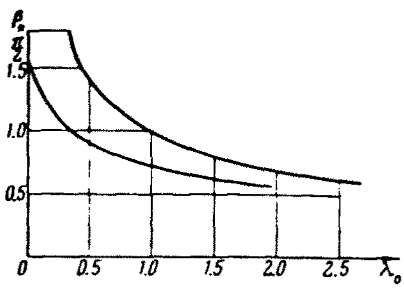

Fig. 4

The presence of a region of stable crack growth in elastic-plastic materials is well known from experiments. In particular, the remarkable experiments of Irwin [1] and McClintock [17] should be mentioned. These experiments carried out on aluminum foil corroborrate the theory presented here quite well.

In conclusion, we note that the results obtained can also be considered valid for plane strain if an approximate model of cracking in an elastic-plastic material is assumed, similar to the brittle model of Leonov and Panasiuk [18]. It should be mentioned that the study of the elastic-plastic Griffith problem was also undertaken by Goodier and Field [19]. However, these authors did not raise the question of the relation of crack length to load.

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